Analysis Of Preventive Intervention Data
Using Mixture Modeling In Mplus

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Mplus Background

• Inefficient dissemination of statistical methods:
  – Many good methods contributions from biostatistics,
    psychometrics, etc are underutilized in practice
• Fragmented presentation of methods:
  – Technical descriptions in many different journals
  – Many different pieces of limited software
• Mplus: Integration of methods in one framework
  – Easy to use: Simple, non-technical language, graphics
  – Powerful: General modeling capabilities
• Mplus versions
• Mplus team: Linda & Bengt Muthén, Thuy Nguyen,
  Tihomir Asparouhov, Michelle Conn
Statistical Concepts Captured By Latent Variables

Continuous Latent Variables
- Measurement errors
- Factors
- Random effects
- Frailties, liabilities
- Variance components
- Missing data

Categorical Latent Variables
- Latent classes
- Clusters
- Finite mixtures
- Missing data

Models That Use Latent Variables

Continuous Latent Variables
- Factor analysis models
- Structural equation models
- Growth curve models
- Multilevel models

Categorical Latent Variables
- Latent class models
- Mixture models
- Discrete-time survival models
- Missing data models

Mplus integrates the statistical concepts captured by latent variables into a general modeling framework that includes not only all of the models listed above but also combinations and extensions of these models.
General Latent Variable Modeling Framework

• Observed variables
  - x background variables (no model structure)
  - y continuous and censored outcome variables
  - u categorical (dichotomous, ordinal, nominal) and count outcome variables

• Latent variables
  - f continuous variables
    - interactions among f’s
  - c categorical variables
    - multiple c’s

Mplus

Several programs in one
• Structural equation modeling
• Item response theory analysis
• Latent class analysis
• Latent transition analysis
• Survival analysis
• Multilevel analysis
• Complex survey data analysis
• Monte Carlo simulation

Fully integrated in the general latent variable framework
### Overview

#### Single-Level Analysis

<table>
<thead>
<tr>
<th>Continuous Observed And Latent Variables</th>
<th>Cross-Sectional</th>
<th>Longitudinal</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Day 1</strong></td>
<td>Regression Analysis</td>
<td><strong>Day 2</strong></td>
</tr>
<tr>
<td></td>
<td>Path Analysis</td>
<td>Growth Analysis</td>
</tr>
<tr>
<td></td>
<td>Exploratory Factor Analysis</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Confirmatory Factor Analysis</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Structural Equation Modeling</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Adding Categorical Observed And Latent Variables</th>
<th>Cross-Sectional</th>
<th>Longitudinal</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Day 3</strong></td>
<td>Regression Analysis</td>
<td><strong>Day 4</strong></td>
</tr>
<tr>
<td></td>
<td>Path Analysis</td>
<td>Latent Transition Analysis</td>
</tr>
<tr>
<td></td>
<td>Exploratory Factor Analysis</td>
<td>Latent Class Growth Analysis</td>
</tr>
<tr>
<td></td>
<td>Confirmatory Factor Analysis</td>
<td>Growth Analysis</td>
</tr>
<tr>
<td></td>
<td>Structural Equation Modeling</td>
<td>Growth Mixture Modeling</td>
</tr>
<tr>
<td></td>
<td>Latent Class Analysis</td>
<td>Discrete-Time Survival</td>
</tr>
<tr>
<td></td>
<td>Factor Mixture Analysis</td>
<td>Mixture Analysis</td>
</tr>
<tr>
<td></td>
<td>Structural Equation Mixture Modeling</td>
<td>Missing Data Analysis</td>
</tr>
</tbody>
</table>

### Overview (Continued)

#### Multilevel Analysis

<table>
<thead>
<tr>
<th>Continuous Observed And Latent Variables</th>
<th>Cross-Sectional</th>
<th>Longitudinal</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Day 5</strong></td>
<td>Regression Analysis</td>
<td><strong>Day 5</strong></td>
</tr>
<tr>
<td></td>
<td>Path Analysis</td>
<td>Growth Analysis</td>
</tr>
<tr>
<td></td>
<td>Exploratory Factor Analysis</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Confirmatory Factor Analysis</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Structural Equation Modeling</td>
<td></td>
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</tbody>
</table>

<table>
<thead>
<tr>
<th>Adding Categorical Observed And Latent Variables</th>
<th>Cross-Sectional</th>
<th>Longitudinal</th>
</tr>
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<tbody>
<tr>
<td><strong>Day 5</strong></td>
<td>Latent Class Analysis</td>
<td><strong>Day 5</strong></td>
</tr>
<tr>
<td></td>
<td>Factor Mixture Analysis</td>
<td>Growth Mixture Modeling</td>
</tr>
</tbody>
</table>
Further Studies

- Mplus web site: www.statmodel.com
  - Short courses
    - Johns Hopkins University, August 20-22, 2007 (twice a year): Multilevel modeling
    - University of Florence, Italy, September 10-12, 2007: Mixture, growth, and multilevel modeling
  - Web videos of courses: 10 weeks, 2 days, 1 day, 2 hours
  - Reference section
  - Paper section (pdf's)
  - Free demo and Mplus User's Guide
  - Mplus Discussion
  - Syllabus, handouts and suggested readings from the UCLA course Statistical Methods for School-Based Intervention Studies, http://www.gseis.ucla.edu/faculty/muthen/courses_Ed255C.htm

Data For Longitudinal Intervention Studies

- Before intervention
  - Baseline individual measures (covariates, measurement instruments)
    - Influencing control group development
    - Influencing treatment group development
    - Influencing dropout
  - Baseline outcomes
- During intervention
  - Outcomes
    - Implementation measures
    - Compliance measures
- After intervention
  - Proximal
  - Distal

<table>
<thead>
<tr>
<th>Before intervention</th>
<th>During intervention</th>
<th>After intervention</th>
</tr>
</thead>
<tbody>
<tr>
<td>Baseline individual measures (covariates, measurement instruments)</td>
<td>Outcomes</td>
<td>Baseline outcomes</td>
</tr>
<tr>
<td>Influencing control group development</td>
<td>Implementation measures</td>
<td>Proximal</td>
</tr>
<tr>
<td>Influencing treatment group development</td>
<td>Compliance measures</td>
<td>Distal</td>
</tr>
</tbody>
</table>
Finding Subgroups By Mixture Modeling

- Baseline data analysis: Latent class analysis

- Analysis of developmental trajectory classes in the absence of intervention
  - Latent transition analysis
  - Growth mixture analysis

- Analysis of developmental trajectory classes in the presence of intervention - for whom is an intervention effective?
Latent Class Analysis

Latent Class Analysis (Continued)

Introduced by Lazarsfeld & Henry, Goodman, Clogg, Dayton & Mcready

- Setting
  - Cross-sectional data
  - Multiple items measuring a construct
  - Hypothesized construct represented as latent class variable (categorical latent variable)

- Aim
  - Identify items that indicate classes well
  - Estimate class probabilities
  - Relate class probabilities to covariates
  - Classify individuals into classes (posterior probabilities)

- Applications
  - Diagnostic criteria for alcohol dependence. National sample, n = 8313
  - Antisocial behavior items measured in the NLSY. National sample, n = 7326
**Latent Class Analysis Model**

Dichotomous (0/1) indicators $u$: $u_1, u_2, \ldots, u_r$

Categorical latent variable $c$: $c = k; k = 1, 2, \ldots, K$.

Marginal probability for item $u_j = 1$,

$$P(u_j = 1) = \sum_{k=1}^{K} P(c = k) P(u_j = 1 | c = k).$$

Joint probability of all $u$’s, assuming conditional independence

$$P(u_1, u_2, \ldots, u_r) = \sum_{k=1}^{K} P(c = k) P(u_1 | c = k) P(u_2 | c = k) \ldots P(u_r | c = k)$$

Note analogies with the case of continuous outcomes and continuous factors

---

**LCA Estimation**

**Posterior Probabilities:**

$$P(c = k | u_1, u_2, \ldots, u_r) = \frac{P(c = k) P(u_1 | c = k) P(u_2 | c = k) \ldots P(u_r | c = k)}{P(u_1, u_2, \ldots, u_r)}$$

**Maximum-likelihood estimation via the EM algorithm:**

$c$ seen as missing data. EM: maximize $E$(complete-data log likelihood $|u_{i1}, u_{i2}, \ldots, u_{ir}$) wrt parameters.

- **E (Expectation) step:** compute $E(c_i | u_{i1}, u_{i2}, \ldots, u_{ir})$ = posterior probability for each class and $E(c_i u_j | u_{i1}, u_{i2}, \ldots, u_{ir})$ for each class and $u_j$

- **M (Maximization) step:** estimate $P(u_j | c_i)$ and $P(c_i)$ parameters by regression and summation over posterior probabilities
Number of $H_0$ parameters in the (exploratory) LCA model with $K$ classes and $r$ binary $u$'s: $K - 1 + K \times r$ ($H_1$ has $2^K - 1$ parameters).

- 2 classes, 3 $u$: $df = 0$ computed as $(8 - 1) - (1 + 6)$
- 2 classes, 4 $u$: $df = 6$ computed as $(16 - 1) - (1 + 8)$
- 3 classes, 4 $u$: $df = 1$, but not identified because of 1 indeterminacy
- 3 classes, 5 $u$: $df = 14$ computed as $(32 - 1) - (2 + 15)$

Confirmatory LCA modeling applies restrictions to the parameters.

**Logit versus Probability Scale.** The $u$-$c$ relation is a logit regression (binary $u$),

\[ P(u = 1 \mid c) = \frac{1 + e^{Logit}}{1 + e^{Logit}}, \quad (81) \]

\[ Logit = \log \left[ \frac{P}{1 - P} \right]. \quad (82) \]

For example:
- Logit = 0: $P = 0.5$
- Logit = -3: $P = 0.05$
- Logit = -1: $P = 0.27$
- Logit = 1: $P = 0.73$

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**LCA Parameters**

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**LCA Testing Against Data**

- Model fit to frequency tables. Overall test against data
  - When the model contains only $u$, summing over the cells,
  \[ \chi^2_P = \sum_i \frac{(o_i - e_i)^2}{e_i}, \quad (82) \]
  \[ \chi^2_{LR} = 2 \sum_i o_i \log \frac{o_i}{e_i}. \quad (83) \]

A cell that has non-zero observed frequency and expected frequency less than .01 is not included in the $\chi^2$ computation as the default. With missing data on $u$, the EM algorithm described in Little and Rubin (1987; chapter 9.3, pp. 181-185) is used to compute the estimated frequencies in the unrestricted multinomial model. In this case, a test of MCAR for the unrestricted model is also provided (Little & Rubin, 1987, pp. 192-193).

- Model fit to univariate and bivariate frequency tables. Mplus TECH10
# Latent Class Analysis

**Alcohol Dependence Criteria, NLSY 1989 (n = 8313)**


<table>
<thead>
<tr>
<th>DSM-III-R Criterion</th>
<th>Conditional Probability of Fulfilling a Criterion</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Two-class solution</td>
</tr>
<tr>
<td></td>
<td>I</td>
</tr>
<tr>
<td>Withdrawal</td>
<td>0.00</td>
</tr>
<tr>
<td>Tolerance</td>
<td>0.01</td>
</tr>
<tr>
<td>Larger</td>
<td>0.15</td>
</tr>
<tr>
<td>Cut down</td>
<td>0.00</td>
</tr>
<tr>
<td>Time spent</td>
<td>0.00</td>
</tr>
<tr>
<td>Major role-Hazard</td>
<td>0.03</td>
</tr>
<tr>
<td>Give up</td>
<td>0.00</td>
</tr>
<tr>
<td>Relief</td>
<td>0.00</td>
</tr>
<tr>
<td>Continue</td>
<td>0.00</td>
</tr>
</tbody>
</table>

1. Likelihood ratio chi-square fit = 1779 with 492 degrees of freedom
2. Likelihood ratio chi-square fit = 448 with 482 degrees of freedom

---

# Latent Class Membership By Number Of DSM-III-R Alcohol Dependence Criteria Met (n=8313)


<table>
<thead>
<tr>
<th>Number of Criteria Met</th>
<th>Two-class solution</th>
<th>Three-class solution</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>I</td>
<td>II</td>
</tr>
<tr>
<td>0</td>
<td>5335</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>1161</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>845</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>469</td>
</tr>
<tr>
<td>4</td>
<td>0</td>
<td>213</td>
</tr>
<tr>
<td>5</td>
<td>0</td>
<td>116</td>
</tr>
<tr>
<td>6</td>
<td>0</td>
<td>68</td>
</tr>
<tr>
<td>7</td>
<td>0</td>
<td>42</td>
</tr>
<tr>
<td>8</td>
<td>0</td>
<td>39</td>
</tr>
<tr>
<td>9</td>
<td>0</td>
<td>24</td>
</tr>
<tr>
<td>%</td>
<td>78.1</td>
<td>21.9</td>
</tr>
</tbody>
</table>
LCA Testing Of K – 1 Versus K Classes

Model testing by $\chi^2$, BIC, and LRT

- Overall test against data: likelihood-ratio $\chi^2$ with $H_1$ as the unrestricted multinomial (problem: sparse cells)
- Comparing models with different number of classes:
  - Likelihood-ratio $\chi^2$ cannot be used
  - Bayesian information criterion (Schwartz, 1978)
    \[
    BIC = -2\log L + h \times \ln n,
    \]
    where $h$ is the number of parameters and $n$ is the sample size. Choose model with smallest BIC value.
  - Vuong-Lo-Mendell-Rubin likelihood-ratio test (Biometrika, 2001). Mplus TECH11
  - Bootstrapped likelihood ratio test. Mplus TECH14 (Version 4)

Other Considerations In Determining The Number Of Classes

Interpretability and usefulness:
- Substantive theory
- Auxiliary (external) variables
- Predictive validity
### LCA Model Results For NLSY Alcohol Dependence Criteria

<table>
<thead>
<tr>
<th>Number of Classes</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pearson $\chi^2$</td>
<td>128,906</td>
<td>773</td>
<td>664</td>
<td>585</td>
</tr>
<tr>
<td>LR $\chi^2$</td>
<td>1,779</td>
<td>448</td>
<td>326</td>
<td>263</td>
</tr>
<tr>
<td>$\chi^2$ df</td>
<td>492</td>
<td>482</td>
<td>472</td>
<td>462</td>
</tr>
<tr>
<td># significant bivariate residuals (TECH10)</td>
<td>84</td>
<td>4</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>Loglikelihood</td>
<td>-14,804</td>
<td>-14,139</td>
<td>-14,078</td>
<td>-14,046</td>
</tr>
<tr>
<td># of parameters</td>
<td>19</td>
<td>29</td>
<td>39</td>
<td>49</td>
</tr>
<tr>
<td>BIC</td>
<td>29,780</td>
<td>28,539</td>
<td>28,508</td>
<td>28,535</td>
</tr>
<tr>
<td>LMR (TECH11) p</td>
<td>0.000</td>
<td>0.000</td>
<td>0.008</td>
<td>0.082</td>
</tr>
<tr>
<td>BLRT (TECH14) p</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
</tr>
<tr>
<td>Entropy</td>
<td>0.901</td>
<td>0.892</td>
<td>0.844</td>
<td>0.852</td>
</tr>
</tbody>
</table>

### LCA Item Profiles For NLSY Alcohol Criteria

#### 2-class LCA Item Profiles

#### 3-class LCA Item Profiles

#### 4-class LCA Item Profiles

#### 5-class LCA Item Profiles
The multinomial logistic regression model expresses the probability that individual $i$ falls in class $k$ of the latent class variable $c$ as a function of the covariate $x$,

$$P(c_i = k | x_i) = \frac{e^{\alpha_k + \gamma_k x_i}}{\sum_{s=1}^{K} e^{\alpha_s + \gamma_s x_i}},$$  \hspace{1cm} (90)

where $\alpha_k = 0$, $\gamma_k = 0$ so that $e^{\alpha_K + \gamma_K x_i} = 1$.

This implies that the log odds comparing class $k$ to the last class $K$ is

$$\log[P(c_i = k | x_i)/P(c_i = K | x_i)] = \alpha_k + \gamma_k x_i.$$  \hspace{1cm} (91)
ASB Classes Regressed On Age, Male, Black In The NLSY (n=7326)

Further Readings On Latent Class Analysis


Latent Class, Factor, And Factor Mixture Analysis
Alcohol Dependence Criteria, NLSY 1989 (n = 8313)


<table>
<thead>
<tr>
<th>DSM-III-R Criterion</th>
<th>Conditional Probability of Fulfilling a Criterion</th>
</tr>
</thead>
<tbody>
<tr>
<td>Withdrawal</td>
<td>0.00     0.14     0.00     0.07     0.49</td>
</tr>
<tr>
<td>Tolerance</td>
<td>0.01     0.45     0.01     0.35     0.81</td>
</tr>
<tr>
<td>Larger</td>
<td>0.15     0.96     0.12     0.94     0.99</td>
</tr>
<tr>
<td>Cut down</td>
<td>0.00     0.14     0.01     0.05     0.60</td>
</tr>
<tr>
<td>Time spent</td>
<td>0.00     0.19     0.00     0.09     0.65</td>
</tr>
<tr>
<td>Major role-Hazard</td>
<td>0.03     0.83     0.02     0.73     0.96</td>
</tr>
<tr>
<td>Give up</td>
<td>0.00     0.10     0.00     0.03     0.43</td>
</tr>
<tr>
<td>Relief</td>
<td>0.00     0.08     0.00     0.02     0.40</td>
</tr>
<tr>
<td>Continue</td>
<td>0.00     0.24     0.02     0.11     0.83</td>
</tr>
</tbody>
</table>

1 Likelihood ratio chi-square fit = 1779 with 492 degrees of freedom
2 Likelihood ratio chi-square fit = 448 with 482 degrees of freedom
LCA, FA, And FMA For NLSY 1989

- LCA, 3 classes: \( \log L = -14,139 \), 29 parameters, BIC = 28,539
- FA, 2 factors: \( \log L = -14,083 \), 26 parameters, BIC = 28,401
- FMA 2 classes, 1 factor, loadings invariant:
  \( \log L = -14,054 \), 29 parameters, BIC = 28,370

Models can be compared with respect to fit to the data

- Standardized bivariate residuals
- Standardized residuals for most frequent response patterns

---

Estimated Frequencies And Standardized Residuals

<table>
<thead>
<tr>
<th>Response Pattern</th>
<th>Obs Freq.</th>
<th>LCA 3c</th>
<th>FA 2f</th>
<th>FMA 1f, 2c</th>
</tr>
</thead>
<tbody>
<tr>
<td>0000000000</td>
<td>5335</td>
<td>5332</td>
<td>-0.07</td>
<td>5307</td>
</tr>
<tr>
<td>0010000000</td>
<td>941</td>
<td>945</td>
<td>0.12</td>
<td>985</td>
</tr>
<tr>
<td>0010010000</td>
<td>601</td>
<td>551</td>
<td>-2.22</td>
<td>596</td>
</tr>
<tr>
<td>0110010000</td>
<td>217</td>
<td>284</td>
<td>4.04</td>
<td>211</td>
</tr>
<tr>
<td>0110000000</td>
<td>155</td>
<td>111</td>
<td>-4.16</td>
<td>118</td>
</tr>
<tr>
<td>0000010000</td>
<td>149</td>
<td>151</td>
<td>0.15</td>
<td>168</td>
</tr>
<tr>
<td>0010010011</td>
<td>65</td>
<td>68</td>
<td>0.41</td>
<td>46</td>
</tr>
<tr>
<td>0100000000</td>
<td>49</td>
<td>52</td>
<td>0.42</td>
<td>84</td>
</tr>
<tr>
<td>0010110000</td>
<td>48</td>
<td>54</td>
<td>0.81</td>
<td>44</td>
</tr>
<tr>
<td>0110010011</td>
<td>47</td>
<td>40</td>
<td>-1.09</td>
<td>45</td>
</tr>
</tbody>
</table>

Bolded entries are significant at the 5% level.

DATA: FILE = bengt05_spread.dat;

VARIABLE: NAMES = u1-u9;
          CATEGORICAL = u1-u9;
          CLASSES = c(2);

ANALYSIS: TYPE = MIXTURE;
          ALGORITHM = INTEGRATION;
          STARTS = 200 10; STITER = 20;
          ADAPTIVE = OFF;
          PROCESS = 4;

MODEL: OVERALL%
        f BY u1-u9;
        f*1; [f@0];

       %c#1%
       [u1$1-u9$1];
        f*1;

       %c#2%
       [u1$1-u9$1];
        f*1;

OUTPUT: TECH1 TECH8 TECH10;

PLOT: TYPE = plot3;
       SERIES = u1-u9(*);
Latent Transition Analysis

- Setting
  - Cross-sectional or longitudinal data
  - Multiple items measuring several different constructs
  - Hypothesized simple structure for measurements
  - Hypothesized constructs represented as latent class variables (categorical latent variables)

- Aim
  - Identify items that indicate classes well
  - Test simple measurement structure
  - Study relationships between latent class variables
  - Estimate class probabilities
  - Relate class probabilities to covariates
  - Classify individuals into classes (posterior probabilities)

- Application
  - Latent transition analysis with four latent class indicators at two time points and a covariate
Latent Transition Analysis

Transition Probabilities

<table>
<thead>
<tr>
<th></th>
<th>c1</th>
<th>c2</th>
</tr>
</thead>
<tbody>
<tr>
<td>c1</td>
<td>0.8</td>
<td>0.2</td>
</tr>
<tr>
<td>c2</td>
<td>0.4</td>
<td>0.6</td>
</tr>
</tbody>
</table>

Time Point 1

\begin{align*}
\text{u11} & \quad \text{u12} & \quad \text{u13} & \quad \text{u14} \\
\text{u21} & \quad \text{u22} & \quad \text{u23} & \quad \text{u24}
\end{align*}

Time Point 2

\begin{align*}
\text{c1} & \quad \text{c2} \\
\text{x}
\end{align*}

Input For LTA With Two Time Points And A Covariate

TITLE: Latent transition analysis for two time points and a covariate

DATA: FILE = mc2tx.dat;

VARIABLE: NAMES ARE u11-u14 u21-u24 x xc1 xc2;
USEV = u11-u14 u21-u24 x;
CATEGORYAL = u11-u24;
CLASSES = c1(2) c2(2);

ANALYSIS: TYPE = MIXTURE;

MODEL: %OVERALL%
  c2#1 ON c1#1 x;
  c1#1 ON x;
Input For LTA With Two Time Points And A Covariate (Continued)

MODEL c1:
  %c1#1%
  [u1$1-u14$1] (1-4);
  %c1#2%
  [u11$1-u14$1] (5-8);

MODEL c2:
  %c2#1%
  [u21$1-u24$1] (1-4);
  %c2#2%
  [u21$1-u24$1] (5-8);

OUTPUT: TECH1 TECH8;

Tests Of Model Fit

Loglikelihood
  H0 Value -3926.187

Information Criteria
  Number of Free Parameters 13
  Akaike (AIC) 7878.374
  Bayesian (BIC) 7942.175
  Sample-Size Adjusted BIC 7900.886
  \( (n^* = (n + 2) / 24) \)
  Entropy 0.902
Chi-Square Test of Model Fit for the Latent Class Indicator Model Part

Pearson Chi-Square
Value  250.298
Degrees of Freedom  244
P-Value  0.3772

Likelihood Ratio Chi-Square
Value  240.811
Degrees of Freedom  244
P-Value  0.5457

Final Class Counts

FINAL CLASS COUNTS AND PROPORTIONS OF TOTAL SAMPLE SIZE BASED ON ESTIMATED POSTERIOR PROBABILITIES

Class 1  328.42644  0.32843
Class 2  184.43980  0.18444
Class 3  146.98726  0.14699
Class 4  340.14650  0.34015

Output Excerpts LTA With Two Time Points And A Covariate (Continued)

Model Results

Estimates S.E. Est./S.E.

Class 1-C1, 1-C2

Thresholds
U1$1  -2.020  0.110 -18.353
U12$1  -2.003  0.106 -18.919
U13$1  -1.776  0.098 -18.046
U14$1  -1.861  0.101 -18.396
U21$1  -2.020  0.110 -18.353
U22$1  -2.003  0.106 -18.919
U23$1  -1.776  0.098 -18.046
U24$1  -1.861  0.101 -18.396

Output Excerpts LTA With Two Time Points And A Covariate (Continued)
### Output Excerpts LTA With Two Time Points And A Covariate (Continued)

#### Class 1-C1, 2-C2

<table>
<thead>
<tr>
<th>Thresholds</th>
<th>Estimates</th>
<th>S.E.</th>
<th>Est./S.E.</th>
</tr>
</thead>
<tbody>
<tr>
<td>U11$1</td>
<td>-2.020</td>
<td>0.110</td>
<td>-18.353</td>
</tr>
<tr>
<td>U12$1</td>
<td>-2.063</td>
<td>0.106</td>
<td>-18.919</td>
</tr>
<tr>
<td>U13$1</td>
<td>-1.741</td>
<td>0.098</td>
<td>-18.046</td>
</tr>
<tr>
<td>U14$1</td>
<td>-1.864</td>
<td>0.101</td>
<td>-18.396</td>
</tr>
<tr>
<td>U21$1</td>
<td>1.964</td>
<td>0.111</td>
<td>17.736</td>
</tr>
<tr>
<td>U22$1</td>
<td>2.107</td>
<td>0.119</td>
<td>18.113</td>
</tr>
<tr>
<td>U23$1</td>
<td>1.864</td>
<td>0.100</td>
<td>18.704</td>
</tr>
<tr>
<td>U24$1</td>
<td>2.107</td>
<td>0.112</td>
<td>18.879</td>
</tr>
</tbody>
</table>

#### Thresholds

- Class 2-C1, 1-C2

#### Estimates S.E. Est./S.E.

#### Class 2-C1, 2-C2

<table>
<thead>
<tr>
<th>Thresholds</th>
<th>Estimates</th>
<th>S.E.</th>
<th>Est./S.E.</th>
</tr>
</thead>
<tbody>
<tr>
<td>U11$1</td>
<td>1.964</td>
<td>0.111</td>
<td>17.736</td>
</tr>
<tr>
<td>U12$1</td>
<td>2.164</td>
<td>0.119</td>
<td>18.113</td>
</tr>
<tr>
<td>U13$1</td>
<td>1.864</td>
<td>0.100</td>
<td>18.704</td>
</tr>
<tr>
<td>U14$1</td>
<td>2.107</td>
<td>0.112</td>
<td>18.879</td>
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<td>U21$1</td>
<td>-2.020</td>
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<td>-18.353</td>
</tr>
<tr>
<td>U22$1</td>
<td>-2.063</td>
<td>0.106</td>
<td>-18.919</td>
</tr>
<tr>
<td>U23$1</td>
<td>-1.741</td>
<td>0.098</td>
<td>-18.046</td>
</tr>
<tr>
<td>U24$1</td>
<td>-1.864</td>
<td>0.101</td>
<td>-18.396</td>
</tr>
</tbody>
</table>
Output Excerpts LTA With Two Time Points And A Covariate (Continued)

<table>
<thead>
<tr>
<th></th>
<th>Estimates</th>
<th>S.E.</th>
<th>Est./S.E.</th>
</tr>
</thead>
<tbody>
<tr>
<td>C2#1 ON</td>
<td>0.530</td>
<td>0.180</td>
<td>2.953</td>
</tr>
<tr>
<td>C1#1</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>C2#1 ON</td>
<td>-1.038</td>
<td>0.107</td>
<td>-9.703</td>
</tr>
<tr>
<td>X</td>
<td>-1.540</td>
<td>0.112</td>
<td>-13.761</td>
</tr>
<tr>
<td>C1#1 ON</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>X</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Intercepts</td>
<td>0.065</td>
<td>0.082</td>
<td>0.797</td>
</tr>
<tr>
<td>C1#1</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>C2#1</td>
<td>-0.407</td>
<td>0.120</td>
<td>-3.381</td>
</tr>
</tbody>
</table>

Further Readings On Latent Transition Analysis


Latent Transition Analysis Extensions

Latent Transition Analysis
And Intervention Studies

Diagram:
- Nodes: u1, u2, c1, c2, Tx
- Edges: u1 to c1, u1 to c2, u2 to c1, u2 to c2, c1 to c2, Tx to c1
VARIABLE:
NAMES = u11-u15 u21-u25 tx;
CATEGORYICAL = u11-u15 u21-u25;
CLASSES = cg(2) c1(2) c2(2);
KNOWNCLASS = cg(tx=0 tx=1);

ANALYSIS:
TYPE = MIXTURE MISSING;
PROCESS = 2;
STARTS = 100 20;

MODEL:
%OVERALL%
c2#1 ON c1#1@0
cg#1 (p0);
c2#1 (p1);

MODEL cg:
%cg#1%
c2#1 ON c1#1 (p2);
cg#2 (p3);
%cg#2%
c2#1 ON c1#1 (p3);

MODEL c1:
%c1#1%
[u11$1-u15$1*1] (1-5);
%c1#2%
[u11$1-u15$1*1] (11-15);

MODEL c2:
%c2#1%
[u21$1-u25$1*1] (1-5);
%c2#2%
[u21$1-u25$1*1] (11-15);

MODEL CONSTRAINT:
NEW(p011 p012 p021 p022 p111 p112 p121 p122 lowlow highlow);
!p0*, p1* contain probabilities for the 4 cells for control and
!intervention groups
!lowlow is the probability effect of intervention on staying in
!the low class
!highlow is the probability effect of intervention on moving from
!high to low class
!the effect is calculated as P(intervention) - P(control)
p011 = exp(p0+p1+p2)/(exp(p0+p1+p2)+1);
p012 = 1/(exp(p0+p1+p2)+1);
p021 = exp(p0+p1)/,(exp(p0*p1)+1);
p022 = 1/(exp(p0+p1)+1);
p111 = exp(p1+p3)/(exp(p1+p3)+1);
p121 = 1/(exp(p1+p3)+1);
p122 = 1/(exp(p1+p3)+1);
lowlow = p111-p011;
highlow = p121-p021;

OUTPUT:
TECH1 PATTERNS;

PLOT:
TYPE = PLOT3;
Factor Mixture Latent Transition Analysis
Muthen (2006)

Factor Mixture Latent Transition Analysis: Aggressive-Disruptive Behavior In The Classroom

- 1,137 first-grade students in Baltimore public schools
- 9 items: Stubborn, Break rules, Break things, Yells at others, Takes others property, Fights, Lies, Teases classmates, Talks back to adults
- Skewed, 6-category items; dichotomized (almost never vs other)
- Two time points: Fall and Spring of Grade 1
- For each time point, a 2-class, 1-factor FMA was found best fitting
Factor Mixture Latent Transition Analysis: Aggressive-Disruptive Behavior In The Classroom (Continued)

<table>
<thead>
<tr>
<th>Model</th>
<th>Loglikelihood</th>
<th># parameters</th>
<th>BIC</th>
</tr>
</thead>
<tbody>
<tr>
<td>Conventional LTA</td>
<td>-8,649</td>
<td>21</td>
<td>17,445</td>
</tr>
<tr>
<td>FMA LTA</td>
<td>-8,012</td>
<td>40</td>
<td>16,306</td>
</tr>
</tbody>
</table>

Factors related across time

Estimated Latent Transition Probabilities, Fall to Spring

<table>
<thead>
<tr>
<th></th>
<th>Conventional LTA</th>
<th>FMA-LTA</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Low</td>
<td>High</td>
</tr>
<tr>
<td>Low</td>
<td>0.93</td>
<td>0.07</td>
</tr>
<tr>
<td>High</td>
<td>0.17</td>
<td>0.83</td>
</tr>
</tbody>
</table>
Growth Modeling

(1) \[ y_{it} = \eta_{0i} + \eta_{1i} x_i + \epsilon_{it} \]

(2a) \[ \eta_{0i} = \alpha_0 + \gamma_0 w_i + \zeta_{0i} \]

(2b) \[ \eta_{1i} = \alpha_1 + \gamma_1 w_i + \zeta_{1i} \]

Individual Development Over Time

\[ y_1 \quad y_2 \quad y_3 \quad y_4 \]

\[ \eta_0 \quad \eta_1 \]

w

i = 1
i = 2
i = 3
Growth Modeling Frameworks/Software

Advantages Of Growth Modeling In A Latent Variable Framework

- Flexible curve shape
- Individually-varying times of observation
- Regressions among random effects
- Multiple processes
- Modeling of zeroes
- Multiple populations
- Multiple indicators
- Embedded growth models
- Categorical latent variables: growth mixtures
Growth Models With Categorical Outcomes

The NIMH Schizophrenia Collaborative Study

- The Data—The NIMH Schizophrenia Collaborative Study (Schizophrenia Data)

  - A group of 64 patients using a placebo and 249 patients on a drug for schizophrenia measured at baseline and at weeks one through six

  - Variables—severity of illness, background variables, and treatment variable

  - Data for the analysis—severity of illness at weeks one, two, four, and six and treatment
Schizophrenia Data: Sample Proportions

Proportion

Week

0.0 0.2 0.4 0.6 0.8 1.0

1 2 3 4 5 6

Placebo Group

Drug Group

illness1

illness2

illness4

illness6

i

s

drug

61

62
Input For Schizophrenia Data Growth Model For Binary Outcomes With A Treatment Variable

TITLE: Schizophrenia Data Growth Model for Binary Outcomes With a Treatment Variable and Scaling Factors
DATA: FILE IS schiz.dat; FORMAT IS 5F1;
VARIABLE: NAMES ARE illness1 illness2 illness4 illness6 drug; ! 0=placebo (n=64) 1=drug (n=249)
CATEGORICAL ARE illness1-illness6;
ANALYSIS: TYPE = MEANSTRUCTURE;
!ESTIMATOR = ML;
MODEL:
i s | illness1@0 illness2@1 illness4@3 illness6@5;
i s ON drug;

Alternative language:
MODEL: i BY illness1-illness6@1;
s BY illness1@0 illness2@1 illness4@3 illness6@5;
[illness1$1 illness2$1 illness4$1 illness6$1] (1);
[s];
i s ON drug;
!(illness1@1 illness2-illness6);

Output Excerpts Schizophrenia Data Growth Model For Binary Outcomes With A Treatment Variable

n = 313

Tests Of Model Fit

Loglikelihood
HO Value -486.337

Information Criteria
Number of Free Parameters 7
Akaike (AIC) 986.674
Bayesian (BIC) 1012.898
Sample-Size Adjusted BIC 990.696
(n* = (n + 2) / 24)

64
### Output Excerpts: Schizophrenia Data Growth Model
For Binary Outcomes With A Treatment Variable (Continued)

<table>
<thead>
<tr>
<th>Selected Estimates</th>
<th>Estimates</th>
<th>S.E.</th>
<th>Est./S.E.</th>
<th>Std</th>
<th>StdYX</th>
</tr>
</thead>
<tbody>
<tr>
<td>I ON DRUG ON S</td>
<td>-0.429</td>
<td>0.825</td>
<td>-0.521</td>
<td>-0.156</td>
<td>-0.063</td>
</tr>
<tr>
<td>I ON DRUG ON S</td>
<td>-0.651</td>
<td>0.259</td>
<td>-2.512</td>
<td>-0.684</td>
<td>-0.276</td>
</tr>
<tr>
<td>I WITH S</td>
<td>-0.925</td>
<td>0.621</td>
<td>-1.489</td>
<td>-0.353</td>
<td>-0.353</td>
</tr>
<tr>
<td>Intercepts I</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
</tr>
<tr>
<td>Thresholds ILLNESS1$1</td>
<td>-5.706</td>
<td>1.047</td>
<td>-5.451</td>
<td></td>
<td></td>
</tr>
<tr>
<td>ILLNESS2$1</td>
<td>-5.706</td>
<td>1.047</td>
<td>-5.451</td>
<td></td>
<td></td>
</tr>
<tr>
<td>ILLNESS4$1</td>
<td>-5.706</td>
<td>1.047</td>
<td>-5.451</td>
<td></td>
<td></td>
</tr>
<tr>
<td>ILLNESS5$1</td>
<td>-5.706</td>
<td>1.047</td>
<td>-5.451</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Residual Variances I</td>
<td>7.543</td>
<td>3.213</td>
<td>2.348</td>
<td>0.996</td>
<td>0.996</td>
</tr>
<tr>
<td>Residual Variances S</td>
<td>0.838</td>
<td>0.343</td>
<td>2.440</td>
<td>0.924</td>
<td>0.924</td>
</tr>
</tbody>
</table>

### Further Readings On Growth Analysis With Categorical Outcomes

Two-Part Growth Modeling:
Frequency Of Heavy Drinking Ages 18 – 25

Olsen and Schafer (2001)

Two-Part Growth Mixture Modeling
Two-Part Modeling Extensions In Mplus

- Growth modeling
  - Distal outcome
  - Parallel processes
  - Trajectory classes (mixtures)
  - Multilevel
- Factor analysis
  - Mixtures
    - Latent classes for binary and continuous parts may be incorrectly picked up as additional factors in conventional analysis
  - Multilevel

Growth Mixture Modeling
Individual Development Over Time

\[ y_{it} = \eta_{0i} + \eta_{1i} x_t + \epsilon_{ti} \]

(2a) \[ \eta_{0i} = \alpha_0 + \gamma_0 w_i + \zeta_{0i} \]

(2b) \[ \eta_{1i} = \alpha_1 + \gamma_1 w_i + \zeta_{1i} \]

Mixtures And Latent Trajectory Classes

Modeling motivated by substantive theories of:

- Multiple Disease Processes: Prostate cancer (Pearson et al.)
- Multiple Pathways of Development: Adolescent-limited versus life-course persistent antisocial behavior (Moffitt), crime curves (Nagin), alcohol development (Zucker, Schulenberg)
- Subtypes: Subtypes of alcoholism (Cloninger, Zucker)
Example: Mixed-Effects Regression Models For Studying The Natural History Of Prostate Disease

![Figure 2. Longitudinal PSA curves estimated from the linear mixed-effects model for the group average (thick solid line) and for each individual in the study (thin solid lines).](image)

Source: Pearson, Morrell, Landis and Carter (1994), *Statistics in Medicine*

---

A Clinical Trial Of Depression Medication: Two-Class Growth Mixture Modeling

![Graph showing placebo non-responders and responders](image)

---
Mplus Graphics For LSAY Math Achievement Trajectory Classes

Poor Development: 20%
Moderate Development: 28%
Good Development: 52%

Dropout: 69% 8% 1%

LSAY Math Achievement Trajectory Classes

Female
Hispanic
Black
Mother’s Ed.
Home Res.
Expectations
Drop Thoughts
Arrested
Expelled

High School Dropout
Growth Mixture Modeling Of Developmental Pathways

• New setting:
  – Sequential, linked processes

• New aims:
  – Using an earlier process to predict a later process
  – Early prediction of failing class

Application: General growth mixture modeling of first- and second-grade reading skills and their Kindergarten precursors; prediction of reading failure (Muthén, Khoo, Francis, Boscardin, 1999). Suburban sample, n = 410.
Assessment Of Reading Skills Development

- Longitudinal multiple-cohort design involving approximately 1000 children with measurements taken four times a year from Kindergarten through grade two (October, December, February, April)
- Grade 1 – Grade 2: reading and spelling skills
- Precursor skills: phonemic awareness (Kindergarten, Grade 1, Grade 2), letters/names/sounds (Kindergarten only), rapid naming
- Standardized reading comprehension tests at the end of Grade 1 and Grade 2 (May).

Three research hypotheses (EARS study; Francis, 1996):
- Kindergarten children will differ in their growth and development in precursor skills
- The rate of development of the precursor skills will relate to the rate of development and the level of attainment of reading and spelling skills — and the individual growth rates in reading and spelling skills will predict performance on standardized tests of reading and spelling
- The use of growth rates for skills and precursors will allow for earlier identification of children at risk for poor academic outcomes and lead to more stable predictions regarding future academic performance

Word Recognition Development In Grades 1 And 2
BIC Curve For Reading Skills Development
**Input For Growth Mixture Model For Reading Skills Development**

**TITLE:** Growth mixture model for reading skills development

**DATA:**
FILE IS newran.dat;

**VARIABLE:** NAMES ARE gender eth wc pa1-pa4 wr1-wr8 l1-l4 s1 r1 s2 r2 rnaming1 rnaming2 rnaming3 rnaming4;
USEVARS = pa1-wr8 rnaming4;
MISSING ARE ALL (999);
CLASSES = c(5);

**ANALYSIS:** TYPE = MIXTURE MISSING;

**MODEL:**

%OVERALL%
   i1 s1 | pa1@-3 pa2@-2 pa3@-1 pa4@0;
   i2 s2 | wr1@-7 wr2@-6 wr3@-5 wr4@-4 wr5@-3 wr6@-2 wr7@-1 wr8@0;
   c#1-c#4 ON rnaming4;

**OUTPUT:** TECH8;

---

**Five Classes Of Reading Skills Development**

<table>
<thead>
<tr>
<th>Kindergarten Growth (Five Classes)</th>
<th>Grades 1 and 2 Growth (Five Classes)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Phonemic Awareness</td>
<td>Word Recognition</td>
</tr>
</tbody>
</table>

Data visualizations showing trends in reading skills development across different classes.
Further Readings On Growth Mixture Modeling


Further Readings On Growth Mixture Modeling (Continued)


Different treatment effects in different trajectory classes


See also Muthen & Curran, 1997 for monotonic treatment effects

• GMM: treatment changes trajectory shape
**Figure 1.** Path Diagrams for Models 1 - 3

**Figure 6.** Estimated Mean Growth Curves and Observed Trajectories for 4-Class model 1 by Class and Intervention Status
Input For Growth Mixtures
In Randomized Trials

TITLE: growth mixtures in randomized trials
DATA: FILE IS toca.dat;
VARIABLE: NAMES ARE sctaa11f sctaa11s sctaa12f sctaa12s sctaa13s sctaa14s sctaa15s sctaa16s sctaa17s intngrp;
MISSING ARE ALL (999);
USEVARIABLES ARE sctaa11f-sctaa17s tx;
CLASSES = c(3);
DEFINE: tx = (intngrp==4);
ANALYSIS: TYPE = MIXTURE MISSING;
MODEL: %OVERALL%
   ac bc qc | sctaa11f@0 sctaa11s@0.5 sctaa12f@1 sctaa12s@1.5 sctaa13s@2.5 sctaa14s@3.5 sctaa15s@4.5 sctaa16s@5.5 sctaa17s@6.5;
   bc qc ON tx;
sctaa11f WITH sctaa11s; sctaa12f WITH sctaa12s;

Input For Growth Mixtures
In Randomized Trials (Continued)

%c#1%
   [ac*3 bc qc]; bc qc ON tx;
%c#2%
   [ac*2 bc qc]; bc qc ON tx;
%c#3%
   [ac*1 bc qc]; bc qc ON tx;
   ac sctaa11f-sctaa17s;
Randomized Trials With Non-Compliance

- Tx group (compliance status observed)
  - Compliers
  - Noncompliers
- Control group (compliance status unobserved)
  - Compliers
  - NonCompliers

Compliers and Noncompliers are typically not randomly equivalent subgroups.

Four approaches to estimating treatment effects:
1. Tx versus Control (Intent-To-Treat; ITT)
2. Tx Compliers versus Control (Per Protocol)
3. Tx Compliers versus Tx NonCompliers + Control (As-Treated)
4. Mixture analysis (Complier Average Causal Effect; CACE):
   - Tx Compliers versus Control Compliers
   - Tx NonCompliers versus Control NonCompliers

CACE: Little & Yau (1998) in Psychological Methods
CACE Estimation Via Mixture Modeling And ML Estimation In Mplus

The latent classes of people are principal strata
Straightforward to add covariates for y and for c. Many extensions possible.

Randomized Trials with NonCompliance: Complier Average Causal Effect (CACE) Estimation

UG Ex 7.23
Ex 7.24
TRAINING DATA

Training data can be used when latent class membership is known for certain individuals in the sample.

Training data must include one variable for each latent class. Each individual receives a value of 0 or 1 for each class variable. A zero indicates that the individual is not allowed to be in the class. A one indicates that the individual is allowed to be in the class.

CACE Application

With CACE models, there are two classes, compliers and noncompliers. The treatment group has known class membership. The control group does not. Therefore, the training data is as follows:

<table>
<thead>
<tr>
<th></th>
<th>Class 1</th>
<th>Class 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Control Group</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>Treatment Group Compliers</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>Treatment Group NonCompliers</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

JOBS Data

The JOBS data are from a Michigan University Prevention Research Center study of interventions aimed at preventing poor mental health of unemployed workers and promoting high quality of reemployment. The intervention consisted of five half-day training seminars that focused on problem solving, decision making group processes, and learning and practicing job search skills. The control group received a booklet briefly describing job search methods and tips. Respondents were recruited from the Michigan Employment Security Commission. After a series of screening procedures, 1801 were randomly assigned to treatment and control conditions. Of the 1249 in the treatment group, only 54% participated in the treatment.

The variables collected in the study include depression scores and outcome measures related to reemployment. Background variables include demographic and psychosocial variables.
Data for the analysis include the outcome variable of depression and the background variables of treatment status, age, education, marital status, SES, ethnicity, a risk score for depression, a pre-intervention depression score, a measure of motivation to participate, and a measure of assertiveness. A subset of 502 individuals classified as having high-risk of depression were analyzed.

The analysis replicates that of Little and Yau (1998).
Complier Average Causal Effect (CACE) estimation in a randomized trial.

FILE IS wjobs.dat;

NAMES ARE depress risk Tx depbase age motivate educ assert single econ nonwhite x10 c1 c2;
USEV ARE depress risk Tx depbase age motivate educ assert single econ nonwhite c1-c2;

CLASSES = c(2);
TRAINING = c1-c2;

%OVERALL%

depress ON Tx risk depbase;
c#1 ON age educ motivate econ assert single nonwhite;

%C#2% !c#2 is the noncomplier class (noshows)
[depress];
depress ON Tx@0;

Tests Of Model Fit

Loglikelihood

H0 Value  -729.414

Information Criteria

Number of Free Parameters  14
Akaike (AIC)  1486.828
Bayesian (BIC)  1545.888
Sample-Size Adjusted BIC  1501.451
(n* = (n + 2) / 24)
Entropy  0.727
### Output Excerpts Complier Average Causal Effect (CACE) Model (Continued)

#### Model Results

**FINAL CLASS COUNTS AND PROPORTIONS OF TOTAL SAMPLE SIZE**

<table>
<thead>
<tr>
<th>Class</th>
<th>Count</th>
<th>Proportion</th>
</tr>
</thead>
<tbody>
<tr>
<td>Class 1</td>
<td>271.9348</td>
<td>0.5417</td>
</tr>
<tr>
<td>Class 2</td>
<td>230.0651</td>
<td>0.4583</td>
</tr>
</tbody>
</table>

**CLASSIFICATION OF INDIVIDUALS BASED ON THEIR MOST LIKELY CLASS MEMBERSHIP**

<table>
<thead>
<tr>
<th>Class</th>
<th>Count</th>
<th>Proportion</th>
</tr>
</thead>
<tbody>
<tr>
<td>Class 1</td>
<td>278</td>
<td>0.5538</td>
</tr>
<tr>
<td>Class 2</td>
<td>224</td>
<td>0.4462</td>
</tr>
</tbody>
</table>

**Average Latent Class Probabilities for Most Likely Latent Class Membership (Row) by Latent Class (Column)**

<table>
<thead>
<tr>
<th>Class 1</th>
<th>Class 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.900</td>
<td>0.100</td>
</tr>
<tr>
<td>0.097</td>
<td>0.903</td>
</tr>
</tbody>
</table>

---

### Output Excerpts Complier Average Causal Effect (CACE) Model (Continued)

#### Model Results (Continued)

<table>
<thead>
<tr>
<th>Class 1</th>
<th>Estimates</th>
<th>S.E.</th>
<th>Est./S.E.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Depress ON</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>TX</td>
<td>-.310</td>
<td>.130</td>
<td>-2.378</td>
</tr>
<tr>
<td>RISK</td>
<td>.912</td>
<td>.247</td>
<td>3.685</td>
</tr>
<tr>
<td>DEPBASE</td>
<td>-1.463</td>
<td>.181</td>
<td>-8.077</td>
</tr>
<tr>
<td>Residual Variances</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>DEPRESS</td>
<td>.506</td>
<td>.037</td>
<td>13.742</td>
</tr>
<tr>
<td>Intercepts</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>DEPRESS</td>
<td>1.812</td>
<td>.299</td>
<td>6.068</td>
</tr>
</tbody>
</table>
### Model Results (Continued)

#### Class 2

<table>
<thead>
<tr>
<th>Depress ON</th>
<th>Estimates</th>
<th>S.E.</th>
<th>Est./S.E.</th>
</tr>
</thead>
<tbody>
<tr>
<td>TX</td>
<td>0.000</td>
<td>0.000</td>
<td>0.00</td>
</tr>
<tr>
<td>RISK</td>
<td>0.912</td>
<td>0.247</td>
<td>3.685</td>
</tr>
<tr>
<td>DEPBASE</td>
<td>-1.463</td>
<td>0.181</td>
<td>-8.077</td>
</tr>
</tbody>
</table>

Residual Variances

| DEPRESS        | .506      | 0.037| 13.742    |

Intercepts

| DEPRESS        | 1.633     | 0.273| 5.977     |

### Output Excerpts Complier Average Causal Effect (CACE) Model (Continued)

#### LATENT CLASS REGRESSION MODEL PART

<table>
<thead>
<tr>
<th>C#1 ON</th>
<th>Estimates</th>
<th>S.E.</th>
<th>Est./S.E.</th>
</tr>
</thead>
<tbody>
<tr>
<td>AGE</td>
<td>0.079</td>
<td>0.015</td>
<td>5.184</td>
</tr>
<tr>
<td>EDUC</td>
<td>0.300</td>
<td>0.068</td>
<td>4.390</td>
</tr>
<tr>
<td>MOTIVATE</td>
<td>0.667</td>
<td>0.157</td>
<td>4.243</td>
</tr>
<tr>
<td>ECON</td>
<td>-0.159</td>
<td>0.152</td>
<td>-1.045</td>
</tr>
<tr>
<td>ASSERT</td>
<td>-0.376</td>
<td>0.143</td>
<td>-2.631</td>
</tr>
<tr>
<td>SINGLE</td>
<td>0.540</td>
<td>0.283</td>
<td>1.908</td>
</tr>
<tr>
<td>NONWHITE</td>
<td>-0.499</td>
<td>0.317</td>
<td>-1.571</td>
</tr>
</tbody>
</table>

Intercepts

| C#1             | -8.740    | 1.590| -5.498    |
Further Readings On
CACE


---

Causal Inference
Causal Inference Concepts

- Potential outcomes
- Principal Stratification
- Finite mixtures

Potential Outcomes Framework

- Treatment variable X (e.g., X dichotomous with X=1 or X=0)
- Observed outcome Y, potential outcome variables Y(1), Y(0)
- Observed outcome under selected trmt x equals potential outcome under trmt assignment X=x : \( y_i = y_i(x) \) if \( x_i = x \)

<table>
<thead>
<tr>
<th>Subject #</th>
<th>X</th>
<th>Y(1)</th>
<th>Y(0)</th>
<th>Causal Effect</th>
<th>Y</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( x_1 = 1 )</td>
<td>( y_1(1) )</td>
<td>( y_1(0) )</td>
<td>( y_1(1) - y_1(0) )</td>
<td>( y_1 = y_1(1) )</td>
</tr>
<tr>
<td>2</td>
<td>( x_2 = 0 )</td>
<td>( y_2(1) )</td>
<td>( y_2(0) )</td>
<td>( y_2(1) - y_2(0) )</td>
<td>( y_2 = y_2(0) )</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>N</td>
<td>( x_N = 1 )</td>
<td>( y_N(1) )</td>
<td>( y_N(0) )</td>
<td>( y_N(1) - y_N(0) )</td>
<td>( y_N = y_N(1) )</td>
</tr>
<tr>
<td>Means</td>
<td></td>
<td></td>
<td></td>
<td>ACE</td>
<td></td>
</tr>
</tbody>
</table>
Causal Inference And Non-Compliance


Angrist, Imbens & Rubin (1996) in JASA
- Conscription into the military randomly allocated via draft lottery

![Diagram](Causal Effects Diagram)

- **Z**: treatment assignment (draft status)
  - \( Z = 1 \): assigned to serve in the military (for low lottery numbers)
  - \( Z = 0 \): not assigned to serve (for high lottery numbers)
- **D**: treatment taken (veteran status)
  - \( D = 1 \): served in the military
  - \( D = 0 \): did not serve in the military
- **Y**: health outcome (mortality after discharge)
- Note that \( D \) is not always = \( Z \)
  - avoid the draft (or deferred for medical reasons); non-compliance: \( Z = 1, D = 0 \)
  - volunteer for military service: \( Z = 0, D = 1 \)

\[
\begin{align*}
\text{Causal Effect Of } Z \text{ On } Y, Y_i(1, D_i(1)) - Y_i(0, D_i(0)), \\
\text{For The Population Of Units Classified By } D_i(0) \text{ And } D_i(1)
\end{align*}
\]

<table>
<thead>
<tr>
<th>( D_i(1) )</th>
<th>( D_i(0) )</th>
<th>( Y_i(1, 0) - Y_i(0, 0) = 0 )</th>
<th>( Y_i(1, 0) - Y_i(0, 1) = 0 )</th>
<th>( Y_i(1, 1) - Y_i(0, 1) = - (Y_i(1) - Y_i(0)) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>( Y_i(1, 0) - Y_i(0, 0) = 0 )</td>
<td>( Y_i(1, 0) - Y_i(0, 1) = - (Y_i(1) - Y_i(0)) )</td>
<td>( Y_i(1, 1) - Y_i(0, 1) = 0 )</td>
</tr>
<tr>
<td>1</td>
<td>Never-taker (( \pi_n ); ( \mu_{1n}, \mu_{0n} ))</td>
<td>Defier (( \pi = 0 ))</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>( Y_i(1, 1) - Y_i(0, 0) = Y_i(1) - Y_i(0) )</td>
<td>( Y_i(1, 1) - Y_i(0, 1) = 0 )</td>
<td>( Y_i(1, 0) - Y_i(0, 0) = 0 )</td>
</tr>
<tr>
<td>0</td>
<td>Complier (( \pi_c ); ( \mu_{1c}, \mu_{0c} ))</td>
<td>Always-taker (( \pi_a ); ( \mu_{1a}, \mu_{0a} ))</td>
<td>0</td>
<td></td>
</tr>
</tbody>
</table>

Average causal effect for compliers

\[
E[(Y_i(1) - Y_i(0)) | D_i(1) - D_i(0) = 1] = \frac{E[Y_i(1,D_i(1)) - Y_i(0,D_i(0))]}{E[D_i(1) - D_i(0)]} (12)
\]

Or, \( \mu_{1c} - \mu_{0c} = (\mu_1 - \mu_0)/\pi_c \)

\( Z \rightarrow Y \) is attributed to \( D \rightarrow Y \) under the exclusion restriction
Causal Effect of D $\rightarrow$ Y Continued

- Mixture of 3 latent classes. Identification of parameters. Mixture means:
  \[ \mu_1 = \pi_c \mu_{1c} + \pi_n \mu_{1n} + \pi_a \mu_{1a} \]
  \[ \mu_0 = \pi_c \mu_{0c} + \pi_n \mu_{0n} + \pi_a \mu_{0a} \]
  \[ \mu_1 - \mu_0 = \pi_c (\mu_{1c} - \mu_{0c}) + \pi_n \times 0 + \pi_a \times 0 \]

- Average causal effect
  \[ \mu_{1c} - \mu_{0c} = (\mu_1 - \mu_0) / \pi_c \]

- Estimated average causal effect
  \[ (\pi_1 - \pi_0) / (p_{c+a} - p_a) \]
  where
  - \( p_{c+a} \) is the proportion in the treatment group who take the treatment
  - \( p_a \) is the proportion in the control group who take the treatment
  In JOBS (Little & Yau, 1998), there are no always-takers (could not get into the seminars if not assigned), so
  \[ p_a = 0 \]
  \[ (\pi_1 - \pi_0) / p_c \]
  which is the Bloom (1984) IV estimate (the less compliance, the more attenuated the treatment and the more you upweight the mean difference).

Analysis With Missing Data
Analysis With Missing Data

Used when individuals are not observed on all outcomes in the analysis to make the best use of all available data and to avoid biases in parameter estimates, standard errors, and tests of model fit.

Types of Missingness

- MCAR -- missing completely at random
  - Variables missing by chance
  - Missing by randomized design
  - Multiple cohorts assuming a single population
- MAR -- missing at random
  - Missingness related to observed variables
  - Missing by selective design
- Non-Ignorable
  - Missingness related to values that would have been observed
  - Missingness related to latent variables

Estimation With Missing Data

Types of Estimation (Little & Rubin, 2002)

- Estimation using listwise deleted sample
  - When MCAR is true, parameter estimates and s.e.’s are consistent but estimates are not efficient
  - When MAR is true but not MCAR, parameter estimates and s.e.’s are not consistent
- Maximum likelihood using all available data
  - When MCAR or MAR is true, parameter estimates and s.e.’s are consistent and estimates are efficient
- Imputation
  - Mean and regression imputation – underestimation of variances and covariances
  - Multiple imputation using all available data – a Bayesian approach – credibility intervals are Bayesian justifiable under MCAR and MAR
- Pattern-mixture – used for non-ignorable missingness
Weighted Least Squares Estimation With Missing Data

Weighted least squares for categorical and censored outcomes

- Assumes MCAR when there are no covariates
- Allows MAR when missingness is a function of covariates

MCAR: Missing By Design
**Two-Cohort Growth Model**

\[
\begin{align*}
\eta_0 & \quad \eta_1 \quad y_\bar{7} \\
\eta_0 & \quad \eta_1 
\end{align*}
\]

\[
\begin{array}{ccccccc}
t = 0 & t = 1 & t = 2 & t = 3 & t = 4 & t = 5 \\
y_1 & y_2 & y_3 & y_4 & y_5 & y_6 \\
y_1 & y_2 & y_3 & y_4 & y_5 & y_6 \\
\end{array}
\]

---

**MAR**

\[
y_i = \alpha + \beta x_i + \zeta_i
\]

\[
E(\zeta) = 0, \quad V(\zeta) = \sigma_\zeta^2
\]

\[
E(x) = \mu_x, \quad V(x) = \sigma_x^2
\]

\[\text{Data Matrix:}\]

\[
\begin{array}{c|c}
\hline
x & y \\
\hline
n_{HL} &  \\
\hline
n_L &  \\
\hline
\end{array}
\]

Complete Data Group

Missing Data
Missing At Random (MAR): 
Missing On y In Bivariate Normal Case

\[ \hat{\mu}_y = \frac{\sum_{i=1}^{n_L+n_H} x_i (n_L + n_H)}{n_L + n_H} = \frac{n_L \bar{x}_L + n_H \bar{x}_H}{n_L + n_H}, \]  
(52)

\[ \hat{\sigma}_{xx} = \sum_{i=1}^{n_L+n_H} \frac{(x_i - \hat{\mu}_x)^2}{n_L + n_H}. \]  
(53)

Consider the regression

\[ y_i = \alpha + \beta x_i + \zeta_i \]  
(54)

estimated by the complete-data (listwise present) sample (sample size \( n_H \))

\[ \hat{\alpha} = \bar{y} - \hat{\beta} \bar{x}, \]  
(55)

\[ \hat{\beta} = \frac{s_{yx}}{s_{xx}}, \]  
(56)

\[ \hat{\sigma}_{\zeta\zeta} = s_{yy} - \hat{\beta}^2 s_{xx}. \]  
(57)

This gives the ML estimates of \( \mu_y \) and \( \sigma_{yy} \), adjusting the complete-data sample statistics:

\[ \hat{\mu}_y = \hat{\alpha} + \hat{\beta} \hat{\mu}_x = \bar{y} + \hat{\beta} (\hat{\mu}_x - \bar{x}), \]  
(58)

\[ \hat{\sigma}_{yy} = \hat{\sigma}_{\zeta\zeta} + \hat{\beta}^2 \hat{\sigma}_{xx} = s_{yy} + \hat{\beta}^2 (\hat{\sigma}_{xx} - s_{xx}). \]  
(59)
Correlates Of Missing Data

- MAR is more plausible when the model includes covariates influencing missing data
- Correlates of missing data may not have a “causal role” in the model, i.e. not influencing dependent variables, in which case including them as covariates can bias model estimates
  - Multiple imputation (Bayes; Schafer, 1997) with two different sets of observed variables
    - Imputation model
    - Analysis model
  - Modeling (ML)
    - Including missing data correlates not as x variables but as “y variables,” freely correlated with all other observed variables

Recent overview in Schafer & Graham (2002).

Missing On X

- Regular modeling concerns the conditional distribution

\[ [y \mid x] \] (1)

that is, as in regular regression the marginal distribution of \([x]\) is not involved. This is fine if there is no missing on x in which case considering

\[ [y \mid x] \]

gives the same estimates as (Joreskog & Goldberger, 1975) considering the joint distribution

\[ [y, x] = [y \mid x] [x] \]
• With missing on $x$, ML under MAR must make a
distributional assumption about $[x]$, typically normality.
The modeling then concerns

$$[y, x] = [y \mid x] \ [x]$$  \hspace{1cm} (2)

which with missing on $[x]$ is an expanded model that makes
stronger assumptions as compared to (1).

• The LHS of (2) shows that $y$ and $x$ are treated the same -
they are both “$y$ variables” in Mplus terminology. This is the
default in Mplus when all $y$’s are continuous. In other cases,
$x$’s can be turned into “$y$’s” e.g. by the model statement

x1-xq;

---

**Technical Aspects Of Ignorable Missing Data: ML Under MAR**

\[
\text{Likelihood: } \sum_{i=1}^{\eta} \log [y_i \mid x_i]. \hspace{1cm} (87)
\]

With missing data on $y$, the $i^{th}$ term of (87) expands into

$$[y_i^{\text{obs}}, y_i^{\text{mis}}, m_i \mid x_i], \hspace{1cm} (88)$$

where $m_i$ is a 0/1 indicator vector of the same length as $y_i$.
The likelihood focuses on the observed variables,

$$[y_i^{\text{obs}}, m_i \mid x_i] = \int [y_i^{\text{obs}}, y_i^{\text{mis}} \mid x_i] \ [m_i \mid y_i^{\text{obs}}, y_i^{\text{mis}}, x_i] \ dy_i^{\text{mis}}, \hspace{1cm} (89)$$

which, when assuming that missingness is not a function of
$y_i^{\text{mis}}$ (that is, assuming MAR),
With distinct parameter sets in (91), the last term can be ignored and maximization can focus on the $[y_i^{obs} | x_i]$ term. This leads to the standard MAR ignorable missing data procedure.

**AMPS Data**

The data are taken from the Alcohol Misuse Prevention Study (AMPS). Forty-nine schools with a total of 2,666 students participated in the study. Students were measured seven times starting in the Fall of Grade 6 and ending in the Spring of Grade 12.

Data for the analysis include the average of three items related to alcohol misuse:

During the past 12 months, how many times did you

- drink more than you planned to?
- feel sick to your stomach after drinking?
- get very drunk?

Responses: (0) never, (1) once, (2) two times, (3) three or more times

Four of the seven timepoints are studied: Fall Grade 6, Spring Grade 6, Spring Grade 7, and Spring Grade 8.
Input For AMPS Growth Model With Missing Data

TITLE: AMPS growth model with missing data
DATA: FILE IS amps.dat;
VARIABLE: NAMES ARE caseid amover0 ovrdnk0 illdrnk0 vrydrn0 amover1 ovrdnk1 illdrnk1 vrydrn1 amover2 ovrdnk2 illdrnk2 vrydrn2 amover3 ovrdnk3 illdrnk3 vrydrn3 amover4 ovrdnk4 illdrnk4 vrydrn4 amover5 ovrdnk5 illdrnk5 vrydrn5 amover6 ovrdnk6 illdrnk6 vrydrn6;
USEV = amover0 amover1 amover2 amover3;
MISSING = ALL (999);
Input For AMPS Growth Model With Missing Data (Continued)

**ANALYSIS:**  TYPE = MISSING H1;

**MODEL:**  
```plaintext
i s | amover0@0 amover1@1 amover2@3 amover3*5;
amover1-amover3  PWITH amover0-amover2;
```

**OUTPUT:**  PATTERNS SAMPSTAT MODINDICES STANDARDIZED;

---

Output Excerpts AMPS Growth Model With Missing Data

**Summary of Data**

Number of patterns  15

SUMMARY OF MISSING DATA PATTERNS

MISSING DATA PATTERNS

<table>
<thead>
<tr>
<th>Pattern</th>
<th>Frequency</th>
<th>Pattern</th>
<th>Frequency</th>
<th>Pattern</th>
<th>Frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>685</td>
<td>6</td>
<td>29</td>
<td>11</td>
<td>104</td>
</tr>
<tr>
<td>2</td>
<td>143</td>
<td>7</td>
<td>11</td>
<td>12</td>
<td>237</td>
</tr>
<tr>
<td>3</td>
<td>73</td>
<td>8</td>
<td>64</td>
<td>13</td>
<td>6</td>
</tr>
<tr>
<td>4</td>
<td>164</td>
<td>9</td>
<td>866</td>
<td>14</td>
<td>1</td>
</tr>
<tr>
<td>5</td>
<td>65</td>
<td>10</td>
<td>208</td>
<td>15</td>
<td>3</td>
</tr>
</tbody>
</table>
COVARIANCE COVERAGE OF DATA
Minimum covariance coverage value  0.100

PROPORTION OF DATA PRESENT

<table>
<thead>
<tr>
<th></th>
<th>AMOVER0</th>
<th>AMOVER1</th>
<th>AMOVER2</th>
<th>AMOVER3</th>
</tr>
</thead>
<tbody>
<tr>
<td>AMOVER0</td>
<td>0.464</td>
<td>0.933</td>
<td>0.753</td>
<td>0.682</td>
</tr>
<tr>
<td>AMOVER1</td>
<td>0.401</td>
<td>0.933</td>
<td>0.715</td>
<td>0.650</td>
</tr>
<tr>
<td>AMOVER2</td>
<td>0.347</td>
<td>0.715</td>
<td>0.753</td>
<td>0.610</td>
</tr>
<tr>
<td>AMOVER3</td>
<td>0.314</td>
<td>0.650</td>
<td>0.610</td>
<td>0.682</td>
</tr>
</tbody>
</table>

Tests Of Model Fit

Chi-square Test of Model Fit

| Value     | 0.011 |
| Degrees of Freedom | 1     |
| P-Value    | 0.9177|

RMSEA (Root Mean Square Error Of Approximation)

| Estimate | 0.000 |
| 90 Percent C.I. | 0.000  0.019 |
| Probability RMSEA <= .05 | 0.997 |
## Output Excerpts AMPS Growth Model With Missing Data (Continued)

### Model Results

<table>
<thead>
<tr>
<th></th>
<th>Estimates</th>
<th>S.E.</th>
<th>Est./S.E.</th>
<th>Std</th>
<th>StdYX</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>A MOVER0</td>
<td>1.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.426</td>
</tr>
<tr>
<td></td>
<td>A MOVER1</td>
<td>1.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.426</td>
</tr>
<tr>
<td></td>
<td>A MOVER2</td>
<td>1.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.426</td>
</tr>
<tr>
<td></td>
<td>A MOVER3</td>
<td>1.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.426</td>
</tr>
<tr>
<td>S</td>
<td>A MOVER0</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
</tr>
<tr>
<td></td>
<td>A MOVER1</td>
<td>1.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.109</td>
</tr>
<tr>
<td></td>
<td>A MOVER2</td>
<td>3.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.327</td>
</tr>
<tr>
<td></td>
<td>A MOVER3</td>
<td>6.244</td>
<td>0.426</td>
<td>14.645</td>
<td>0.680</td>
</tr>
<tr>
<td>S I</td>
<td>A MOVER1 WITH A MOVER0</td>
<td>-0.007</td>
<td>0.003</td>
<td>-2.278</td>
<td>-0.146</td>
</tr>
<tr>
<td></td>
<td>A MOVER2 WITH A MOVER0</td>
<td>-0.022</td>
<td>0.011</td>
<td>-2.010</td>
<td>-0.022</td>
</tr>
<tr>
<td></td>
<td>A MOVER2 WITH A MOVER1</td>
<td>0.017</td>
<td>0.007</td>
<td>2.505</td>
<td>0.017</td>
</tr>
</tbody>
</table>
|          | A MOVER3 WITH A MOVER2  | -0.001| 0.027     | -0.050 | -0.001 | -0.003 | 139

### Residual Variances

<table>
<thead>
<tr>
<th></th>
<th>Residual Variances</th>
</tr>
</thead>
<tbody>
<tr>
<td>A MOVER0</td>
<td>0.033 0.013 2.509 0.033 0.152</td>
</tr>
<tr>
<td>A MOVER1</td>
<td>0.123 0.011 10.950 0.123 0.406</td>
</tr>
<tr>
<td>A MOVER2</td>
<td>0.190 0.017 11.461 0.190 0.433</td>
</tr>
<tr>
<td>A MOVER3</td>
<td>0.091 0.068 1.340 0.091 0.140</td>
</tr>
</tbody>
</table>

### Variances

|          | I 0.182 0.014 12.891 1.000 1.000 |
|----------| S 0.012 0.002 5.378 1.000 1.000 |

### Means

|          | I 0.200 0.010 19.391 0.469 0.469 |
|----------| S 0.057 0.005 11.858 0.520 0.520 |

### Intercept

|          | A MOVER0 0.000 0.000 0.000 0.000 0.000 |
|----------| A MOVER1 0.000 0.000 0.000 0.000 0.000 |
|          | A MOVER2 0.000 0.000 0.000 0.000 0.000 |
|          | A MOVER3 0.000 0.000 0.000 0.000 0.000 |
Output Excerpts AMPS Growth Model With Missing Data (Continued)

R-SQUARE

<table>
<thead>
<tr>
<th>Observed Variable</th>
<th>R-Square</th>
</tr>
</thead>
<tbody>
<tr>
<td>AMOVER0</td>
<td>0.848</td>
</tr>
<tr>
<td>AMOVER1</td>
<td>0.594</td>
</tr>
<tr>
<td>AMOVER2</td>
<td>0.567</td>
</tr>
<tr>
<td>AMOVER3</td>
<td>0.860</td>
</tr>
</tbody>
</table>

AMPS: Estimated Growth Curves
Growth Mixture Modeling
With Ignorable Missingness

Selection modeling: \([y \mid x] [m \mid y, x]\). Different approaches to \([m \mid y, x]\):

- Diggle & Kenward (1994) in Applied Statistics:
  - using \(y, y^*\) (non-ignorable dropout)
- Wu & Carroll (1988), Wu & Bailey (1989) in Biometrics:
  - using the slope \(s\)
- Frangakis & Rubin (1999) in Biometrika:
  - using a latent class variable \(c\) (compliance)
- Muthen, Jo, Brown (2003) in JASA:
  - using \(c\) and \(s\) (GMM)

Pattern-mixture modeling: \([m \mid x] [y \mid m, x]\)

- Little & Rubin (2002): overview
- Roy (2003) in Biometrics:
  - using a latent class variable \(c\) (missing data patterns)

Non-Ignorable Missing Data
Modeling Approaches And References
Growth Mixture Modeling With Non-Ignorable Missingness As A Function Of y

Outcome
Escalating
Early Onset
Normative Age

Growth Mixture Modeling With Non-Ignorable Missingness As A Function Of s

Outcome
Escalating
Early Onset
Normative Age

Growth Mixture Modeling With Non-Ignorable Missingness As A Function Of \( c \)
Further Readings On Missing Data Analysis


Multilevel Growth Models
Growth Modeling Approached In Two Ways: Data Arranged As Wide Versus Long

- Wide: Multivariate, Single-Level Approach
  \[ y_{it} = i_t + s_t \times \text{time}_{it} + \epsilon_{it} \]
  - \( i_t \) regressed on \( w_i \)
  - \( s_t \) regressed on \( w_i \)

- Long: Univariate, 2-Level Approach (CLUSTER = id)
  - Within
  - Between

  The intercept \( i \) is called \( y \) in Mplus

Growth Modeling Approached In Two Ways: Data Arranged As Wide Versus Long (Continued)

- Wide (one person):
  \[ \text{t1} \quad \text{t2} \quad \text{t3} \quad \text{t1} \quad \text{t2} \quad \text{t3} \]
  
  Person i:
  \[ \text{id} \quad \text{y1} \quad \text{y2} \quad \text{y3} \quad \text{x1} \quad \text{x2} \quad \text{x3} \quad \text{w} \]

- Long (one cluster):
  \[ \text{t1} \quad \text{id} \quad \text{y1} \quad \text{x1} \quad \text{w} \]
  \[ \text{t2} \quad \text{id} \quad \text{y2} \quad \text{x2} \quad \text{w} \]
  \[ \text{t3} \quad \text{id} \quad \text{y3} \quad \text{x3} \quad \text{w} \]
Three-Level Modeling In Multilevel Terms

Time point $t$, individual $i$, cluster $j$.

$y_{ij}$ : individual-level, outcome variable
$a_{1ij}$ : individual-level, time-related variable (age, grade)
$a_{2ij}$ : individual-level, time-varying covariate
$x_{ij}$ : individual-level, time-invariant covariate
$w_j$ : cluster-level covariate

Three-level analysis (Mplus considers Within and Between)

**Level 1 (Within):**

$y_{ij} = \pi_{0ij} + \pi_{1ij} a_{1ij} + \pi_{2ij} a_{2ij} + e_{ij}$, \hspace{1cm} (1)

$$
\begin{align*}
\pi_{0ij} &= \beta_{00j} + \gamma_{001} w_j + u_{00j}, \\
\pi_{1ij} &= \beta_{10j} + \gamma_{101} w_j + u_{10j}, \\
\pi_{2ij} &= \beta_{20j} + \gamma_{201} w_j + u_{20j}.
\end{align*}
$$

**Level 2 (Within):**

$$
\begin{align*}
\pi_{0ij} &= \beta_{00j} + \gamma_{001} w_j + u_{00j}, \\
\pi_{1ij} &= \beta_{10j} + \gamma_{101} w_j + u_{10j}, \\
\pi_{2ij} &= \beta_{20j} + \gamma_{201} w_j + u_{20j}.
\end{align*}
$$

**Level 3 (Between):**

$$
\begin{align*}
\beta_{00j} &= \gamma_{000} + \gamma_{001} w_j + u_{00j}, \\
\beta_{10j} &= \gamma_{100} + \gamma_{101} w_j + u_{10j}, \\
\beta_{20j} &= \gamma_{200} + \gamma_{201} w_j + u_{20j}, \\
\beta_{0ij} &= \gamma_{010} + \gamma_{011} w_j + u_{0ij}, \\
\beta_{1ij} &= \gamma_{110} + \gamma_{111} w_j + u_{1ij}, \\
\beta_{2ij} &= \gamma_{210} + \gamma_{211} w_j + u_{2ij}.
\end{align*}
$$

Two-Level Growth Modeling
(Three-Level Modeling)

Within

- $y_1$
- $y_2$
- $y_3$
- $y_4$

Between

- $y_1$
- $y_2$
- $y_3$
- $y_4$

$x$

$w$

$iw$

$sw$

$ib$

$sb$
LSAY Two-Level Growth Model

Within

math7 → iw → sw → math8 → sb → math9 → sb → math10 → sb → motheds → homeres

Between

math7 → motheds → homeres

Input For LSAY Two-Level Growth Model With Free Time Scores And Covariates

TITLE: LSAY two-level growth model with free time scores and covariates

DATA: FILE IS lsay98.dat;
FORMAT IS 3f8 f8.4 8f8.2 3f8 2f8.2;
VARIABLE: NAMES ARE cohort id school weight math7 math8 math9 math10 att7 att8 att9 att10 gender motheds homeres;
USEOBS = (gender EQ 1 AND cohort EQ 2);
MISSING = ALL (999);
USEVAR = math7-math10 motheds homeres;
CLUSTER = school;
ANALYSIS: TYPE = TWOLEVEL;
ESTIMATOR = MUML;
Input For LSAy Two-Level Growth Model With Free Time Scores And Covariates (Continued)

MODEL:  
%WITHIN%
iw sw | math7@0 math8@1
math9*2 (1)
math10*3 (2);
iw sw ON mothed homeres;

%BETWEEN%
ib sb | math7@0 math8@1
math9*2 (1)
math10*3 (2);
ib sb ON mothed homeres;

OUTPUT SAMPSTAT STANDARDIZED RESIDUAL;

Output Excerpts LSAy Two-Level Growth Model With Free Time Scores And Covariates

Summary of Data

Number of clusters  50
Size (s) Cluster ID with Size s
1     114
2     136
6     132    304
34    104
39    309
40    302

Average cluster size 18.627

Estimated Intraclass Correlations for the Y Variables

<table>
<thead>
<tr>
<th>Variable</th>
<th>Intraclass Correlation</th>
<th>Variable</th>
<th>Intraclass Correlation</th>
<th>Variable</th>
<th>Intraclass Correlation</th>
</tr>
</thead>
<tbody>
<tr>
<td>MATH7</td>
<td>0.199</td>
<td>MATH8</td>
<td>0.149</td>
<td>MATH9</td>
<td>0.168</td>
</tr>
<tr>
<td>MATH10</td>
<td>0.165</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Tests Of Model Fit

Chi-square Test of Model Fit
Value 24.058*
Degrees of Freedom 14
P-Value 0.0451
CFI / TLI
CFI 0.997
TLI 0.995
RMSEA (Root Mean Square Error Of Approximation)
Estimate 0.028
SRMR (Standardized Root Mean Square Residual)
Value for Between 0.048
Value for Within 0.007

Model Results

Within Level

<table>
<thead>
<tr>
<th>Variable</th>
<th>SW</th>
<th>BY</th>
<th>MATH8</th>
<th>MATH9</th>
<th>MATH10</th>
<th>SW</th>
<th>BY</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td>1.000</td>
<td>2.487</td>
<td>3.589</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>0.000</td>
<td>0.163</td>
<td>0.223</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>0.000</td>
<td>15.220</td>
<td>16.076</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>1.073</td>
<td>2.670</td>
<td>3.853</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>0.128</td>
<td>0.288</td>
<td>0.368</td>
<td></td>
<td></td>
</tr>
<tr>
<td>IW</td>
<td>ON</td>
<td>MOTHED</td>
<td>1.780</td>
<td>0.232</td>
<td>7.665</td>
<td>0.246</td>
<td>0.226</td>
</tr>
<tr>
<td></td>
<td></td>
<td>HOMERES</td>
<td>0.892</td>
<td>0.221</td>
<td>4.031</td>
<td>0.124</td>
<td>0.173</td>
</tr>
<tr>
<td>SW</td>
<td>ON</td>
<td>MOTHED</td>
<td>0.053</td>
<td>0.063</td>
<td>0.836</td>
<td>0.049</td>
<td>0.045</td>
</tr>
<tr>
<td></td>
<td></td>
<td>HOMERES</td>
<td>0.135</td>
<td>0.044</td>
<td>3.047</td>
<td>0.125</td>
<td>0.176</td>
</tr>
</tbody>
</table>
### Output Excerpts LSAY Two-Level Growth Model With Free Time Scores And Covariates (Continued)

#### SW WITH

<table>
<thead>
<tr>
<th></th>
<th>Estimates</th>
<th>S.E.</th>
<th>Est./S.E.</th>
<th>Std</th>
<th>StdYX</th>
</tr>
</thead>
<tbody>
<tr>
<td>IW</td>
<td>2.112</td>
<td>0.522</td>
<td>4.044</td>
<td>0.273</td>
<td>0.273</td>
</tr>
<tr>
<td>HOMERES WITH</td>
<td>0.261</td>
<td>0.039</td>
<td>6.709</td>
<td>0.261</td>
<td>0.203</td>
</tr>
</tbody>
</table>

#### Residual Variances

<table>
<thead>
<tr>
<th></th>
<th>Estimates</th>
<th>S.E.</th>
<th>Est./S.E.</th>
<th>Std</th>
<th>StdYX</th>
</tr>
</thead>
<tbody>
<tr>
<td>MATH7</td>
<td>12.748</td>
<td>1.434</td>
<td>8.888</td>
<td>12.748</td>
<td>0.197</td>
</tr>
<tr>
<td>MATH8</td>
<td>12.298</td>
<td>0.893</td>
<td>13.771</td>
<td>12.298</td>
<td>0.174</td>
</tr>
<tr>
<td>MATH9</td>
<td>14.237</td>
<td>1.132</td>
<td>12.578</td>
<td>14.237</td>
<td>0.166</td>
</tr>
<tr>
<td>MATH10</td>
<td>24.829</td>
<td>2.230</td>
<td>11.133</td>
<td>24.829</td>
<td>0.226</td>
</tr>
<tr>
<td>IW</td>
<td>47.060</td>
<td>3.069</td>
<td>15.333</td>
<td>0.903</td>
<td>0.903</td>
</tr>
<tr>
<td>SW</td>
<td>1.110</td>
<td>0.286</td>
<td>3.879</td>
<td>0.964</td>
<td>0.964</td>
</tr>
</tbody>
</table>

#### MOTHED WITH

<table>
<thead>
<tr>
<th></th>
<th>Estimates</th>
<th>S.E.</th>
<th>Est./S.E.</th>
<th>Std</th>
<th>StdYX</th>
</tr>
</thead>
<tbody>
<tr>
<td>HOMERES</td>
<td>1.970</td>
<td>0.069</td>
<td>28.643</td>
<td>1.970</td>
<td>1.000</td>
</tr>
</tbody>
</table>

### Output Excerpts LSAY Two-Level Growth Model With Free Time Scores And Covariates (Continued)

#### Between Level

<table>
<thead>
<tr>
<th></th>
<th>Estimates</th>
<th>S.E.</th>
<th>Est./S.E.</th>
<th>Std</th>
<th>StdYX</th>
</tr>
</thead>
<tbody>
<tr>
<td>SB</td>
<td>BY</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>MATH8</td>
<td>1.000</td>
<td>0.000</td>
<td>1.000</td>
<td>0.196</td>
<td>0.052</td>
</tr>
<tr>
<td>MATH9</td>
<td>2.487</td>
<td>0.163</td>
<td>15.220</td>
<td>0.488</td>
<td>0.119</td>
</tr>
<tr>
<td>MATH10</td>
<td>3.589</td>
<td>0.223</td>
<td>16.076</td>
<td>0.704</td>
<td>0.115</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>Estimates</th>
<th>S.E.</th>
<th>Est./S.E.</th>
<th>Std</th>
<th>StdYX</th>
</tr>
</thead>
<tbody>
<tr>
<td>IB ON MOTHED</td>
<td>-1.225</td>
<td>2.587</td>
<td>-0.474</td>
<td>-0.362</td>
<td>-0.107</td>
</tr>
<tr>
<td>HOMERES</td>
<td>7.160</td>
<td>1.847</td>
<td>3.876</td>
<td>2.117</td>
<td>1.011</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>Estimates</th>
<th>S.E.</th>
<th>Est./S.E.</th>
<th>Std</th>
<th>StdYX</th>
</tr>
</thead>
<tbody>
<tr>
<td>SB ON MOTHED</td>
<td>0.995</td>
<td>0.647</td>
<td>1.538</td>
<td>5.073</td>
<td>1.493</td>
</tr>
<tr>
<td>HOMERES</td>
<td>0.017</td>
<td>0.373</td>
<td>0.045</td>
<td>0.086</td>
<td>0.041</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>Estimates</th>
<th>S.E.</th>
<th>Est./S.E.</th>
<th>Std</th>
<th>StdYX</th>
</tr>
</thead>
<tbody>
<tr>
<td>SB WITH IB</td>
<td>0.382</td>
<td>0.248</td>
<td>1.538</td>
<td>0.575</td>
<td>0.575</td>
</tr>
</tbody>
</table>
Output Excerpts LSAy Two-Level Growth Model With Free Time Scores And Covariates (Continued)

### Residual Variances

<table>
<thead>
<tr>
<th>Variable</th>
<th>Residual Variances</th>
</tr>
</thead>
<tbody>
<tr>
<td>MATH7</td>
<td>2.059 0.552 3.732 2.059 0.153</td>
</tr>
<tr>
<td>MATH8</td>
<td>0.544 0.268 2.033 0.544 0.039</td>
</tr>
<tr>
<td>MATH9</td>
<td>0.105 0.213 0.493 0.105 0.006</td>
</tr>
<tr>
<td>MATH10</td>
<td>1.395 0.504 2.767 1.395 0.067</td>
</tr>
<tr>
<td>IB</td>
<td>1.428 1.690 0.845 0.125 0.125</td>
</tr>
<tr>
<td>SB</td>
<td>-0.051 0.071 -0.713 -1.321 -1.321</td>
</tr>
</tbody>
</table>

### Variances

<table>
<thead>
<tr>
<th>Variable</th>
<th>Variances</th>
</tr>
</thead>
<tbody>
<tr>
<td>MOTHED</td>
<td>0.087 0.023 3.801 0.087 1.000</td>
</tr>
<tr>
<td>HOMERES</td>
<td>0.228 0.056 4.066 0.228 1.000</td>
</tr>
</tbody>
</table>

### Means

<table>
<thead>
<tr>
<th>Variable</th>
<th>Means</th>
</tr>
</thead>
<tbody>
<tr>
<td>MOTHED</td>
<td>2.307 0.043 53.277 2.307 7.838</td>
</tr>
<tr>
<td>HOMERES</td>
<td>3.108 0.062 50.375 3.108 6.509</td>
</tr>
</tbody>
</table>

### Intercepts

<table>
<thead>
<tr>
<th>Variable</th>
<th>Intercepts</th>
</tr>
</thead>
<tbody>
<tr>
<td>IB</td>
<td>33.510 2.678 12.512 9.909 9.909</td>
</tr>
<tr>
<td>SB</td>
<td>0.163 0.776 0.210 0.830 0.830</td>
</tr>
</tbody>
</table>

### R-Square

#### Within Level

<table>
<thead>
<tr>
<th>Observed Variable</th>
<th>R-Square</th>
</tr>
</thead>
<tbody>
<tr>
<td>MATH7</td>
<td>0.803</td>
</tr>
<tr>
<td>MATH8</td>
<td>0.826</td>
</tr>
<tr>
<td>MATH9</td>
<td>0.834</td>
</tr>
<tr>
<td>MATH10</td>
<td>0.774</td>
</tr>
</tbody>
</table>

#### Latent Variable

<table>
<thead>
<tr>
<th>Latent Variable</th>
<th>R-Square</th>
</tr>
</thead>
<tbody>
<tr>
<td>IW</td>
<td>0.097</td>
</tr>
<tr>
<td>SW</td>
<td>0.036</td>
</tr>
</tbody>
</table>
Output Excerpts LSA Y Two-Level Growth Model With Free Time Scores And Covariates (Continued)

R-Square
Between Level

<table>
<thead>
<tr>
<th>Observed Variable</th>
<th>R-Square</th>
</tr>
</thead>
<tbody>
<tr>
<td>MATH7</td>
<td>0.847</td>
</tr>
<tr>
<td>MATH8</td>
<td>0.961</td>
</tr>
<tr>
<td>MATH9</td>
<td>0.994</td>
</tr>
<tr>
<td>MATH10</td>
<td>0.933</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Latent Variable</th>
<th>R-Square</th>
</tr>
</thead>
<tbody>
<tr>
<td>IW</td>
<td>0.875</td>
</tr>
<tr>
<td>SW</td>
<td>Undefined 0.23207E+01</td>
</tr>
</tbody>
</table>

Further Readings On Three-Level Growth Modeling


**Multilevel Modeling With A Random Slope For Latent Variables**

**Student (Within)**

**School (Between)**

**Two-Level, Two-Part Growth Modeling**

**Within**

**Between**

Multilevel Mixture Modeling

Two-Level Regression Mixture Modeling: Cluster-Randomized CACE
Two-Level Latent Class Analysis

Within

\[ c \]

\[ u_1 \quad u_2 \quad u_3 \quad u_4 \quad u_5 \quad u_6 \]

\[ x \]

Between

\[ c\#1 \]

\[ c\#2 \]

\[ f \]

\[ w \]

Asparouhov & Muthen (2006)

Two-Level Latent Transition Analysis

Within

\[ c_1 \]

\[ u_{11} \ldots u_{1p} \]

\[ c_1\#1 \]

Between

\[ c_2 \]

\[ u_{21} \ldots u_{2p} \]

\[ c_2\#2 \]
Multilevel Growth Mixture Modeling

Growth Mixture Modeling: LSAY Math Achievement Trajectory Classes And The Prediction Of High School Dropout

Poor Development: 20%  Moderate Development: 28%  Good Development: 52%

Dropout: 69%  8%  1%
Input For A Multilevel Growth Mixture Model For LSAY Math Achievement

TITLE: multilevel growth mixture model for LSAY math achievement

DATA: FILE = lsayfull_Dropout.dat;

VARIABLE: NAMES = female mothed homeres math7 math8 math9 math10 expel arrest hisp black hsdrop expect lunch mstrat droptht7 schcode;
!lunch = % of students eligible for full lunch !assistance (9th)
!mstrat = ratio of students to full time math !teachers (9th)
MISSING = ALL (9999);
CATEGORICAL = hsdrop;
CLASSES = c (3);
CLUSTER = schcode;
WITHIN = female mothed homeres expect droptht7 expel arrest hisp black;
BETWEEN = lunch mstrat;
DEFINE: lunch = lunch/100;
mstrat = mstrat/1000;

ANALYSIS: TYPE = MIXTURE TWOLEVEL MISSING;
ALGORITHM = INTEGRATION;

OUTPUT: SAMPSTAT STANDARDIZED TECH1 TECH8;

PLOT: TYPE = PLOT3;
SERIES = math7-math10 (s);

MODEL: %WITHIN%

%OVERALL%

i s | math7@0 math8@1 math9@2 math10@3;
i s ON female hisp black mothed homeres expect
dropht7 expel arrest;
c#1 c#2 ON female hisp black mothed homeres expect
dropht7 expel arrest;
hsdrop ON female hisp black mothed homeres expect
dropht7 expel arrest;
Input For A Multilevel Growth Mixture Model
For LSAY Math Achievement (Continued)

%c1%
[i 40 s 1];
math7=math10*20;
i*13 s 3;
%c2%
[i 40 s 5];
math7=math10*30;
i*8 s 3;
 i s ON female hisp black mothed homeres expect
droptht7 expel arrest;
%c3%
[i 45 s 3];
math7=math10*10;
i*4 s 2;
 i s ON female hisp black mothed homeres expect
droptht7 expel arrest;

%BETWEEN%
%OVERALL%
ib | math7=math10@1; [ib@0];
ib*1; hsdrop*1; ib WITH hsdrop;
math7=math10@0;
ib ON lunch mstrat;
c#1 c#2 ON lunch mstrat;
hsdrop ON lunch mstrat;
%c1%
[hsdrop$1*-.3];
%c2%
[hsdrop$1*.9];
%c3%
[hsdrop$1*1.2];
Power For Growth Models

Designing Future Studies: Power

- Computing power for growth models using Satorra-Saris (Muthén & Curran, 1997; examples)
- Computing power using Monte Carlo studies (Muthén & Muthén, 2002)
- Power calculation web site – PSMG
- Multilevel power (Miyazaki & Raudenbush, 2000; Moerbeek, Breukelen & Berger, 2000; Raudenbush, 1997; Raudenbush & Liu, 2000)
- School-based studies (Brown & Liao, 1999: Principles for designing randomized preventive trials)
- Multiple- (sequential-) cohort power
- Designs for follow-up (Brown, Indurkhia, & Kellam, 2000)
Designing Future Studies: Power

\[ \chi^2 \]

H₀ correct

Type I error

5%

H₀ incorrect

Type II error

P (Rejecting | H₀ incorrect) = Power

Power Estimation For Growth Models Using Monte Carlo Studies

Muthén & Muthén (2002)
TITLE:  This is an example of a Monte Carlo simulation study for a linear growth model for a continuous outcome with missing data where attrition is predicted by time-invariant covariates (MAR)

MONTECARLO:  NAMES ARE y1-y4 x1 x2;
NOSERATIONS = 500;
NREPS = 500;
SEED = 4533;
CUTPOINTS = x2(1);
MISSING = y1-y4;

Input Power Estimation For Growth Models Using Monte Carlo Studies (Continued)

MODEL POPULATION:  x1-x2@1;
[x1-x2@0];
i s | y1@0 y2@1 y3@2 y4@3;
[i*1 s*2];
i*1; s*.2; i WITH s*.1;
y1=y4*.5;
i ON x1*.8 x2*.5;
s ON x1*.4 x2*.25;

MODEL MISSING:  [y1-y4@1];
y1 ON x1*.4 x2*.2;
y2 ON x1*.8 x2*.4;
y3 ON x1*1.6 x2*.8;
y4 ON x1*3.2 x2*1.6;
Input Power Estimation For Growth Models Using Monte Carlo Studies (Continued)

ANALYSIS: TYPE = MISSING H1;
MODEL: i s | y1@0 y2@1 y3@2 y4@3;
[i*1 s*2];
i*1; s*.2; i WITH s*.1;
y1-y4*.5;
i ON x1*1 x2*.5;
s ON x1*.4 x2*.25;
OUTPUT: TECH9;

Output Excerpts Power Estimation For Growth Models Using Monte Carlo Studies

Model Results

<table>
<thead>
<tr>
<th></th>
<th>ESTIMATES</th>
<th>S.E.</th>
<th>M. S. E.</th>
<th>95%</th>
<th>%Sig</th>
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<tr>
<td></td>
<td>Population</td>
<td>Average</td>
<td>Std. Dev.</td>
<td>Average</td>
<td>Cover Coeff</td>
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<td></td>
</tr>
<tr>
<td>X1</td>
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<td>1.0032</td>
<td>0.0598</td>
<td>0.0579</td>
<td>0.0036</td>
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<td>0.5076</td>
<td>0.1554</td>
<td>0.1570</td>
<td>0.0241</td>
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<tr>
<td>S ON</td>
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<td></td>
</tr>
<tr>
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<td>0.2469</td>
<td>0.0865</td>
<td>0.0877</td>
<td>0.0075</td>
</tr>
</tbody>
</table>
References


References (continued)


References (continued)


On Categorical Data Analysis:


On Growth Modeling: